ECS 332: Solution for Problem Set 3

Problem 1

a. Give a ***simplified*** expression for the Fourier transform $P(f)$ of a waveform $p(t)$ when

$$p(t)=\begin{cases} A, & 0\le t<T \\ 0, & \text{otherwise}\end{cases}$$

Solution

$$P(f)=\int_{-\infty}^{\infty} c(t)e^{-j2\pi ft}dt=\int_{0}^{T} Ae^{-j2\pi ft}dt=A\frac{1}{-j2\pi f}e^{-j2\pi ft}\Big|_{0}^{T}$$

$$=A\frac{1}{-j2\pi f}\left(e^{-j2\pi fT}-1\right)=\frac{A}{j2\pi f}\left(1-e^{-j2\pi fT}\right)$$

b. A message $m=\left(m[0],m[1],m[2],m[3]\right)=(1,-1,1,1)$ is sent via

$$x(t)=\sum_{k=0}^{\ell-1} m[k]c(t-kT) \text{ where } \ell \text{ is the length of } m.$$

Find a ***simplified*** expression for the Fourier transform $X(f)$ of the waveform $x(t)$.

Solution

We start with $x(t)=\sum_{k=0}^{\ell-1} m[k]c(t-kT)\xrightarrow{\mathcal{F}}X(f)=P(f)\sum_{k=0}^{\ell-1} m[k]e^{-j2\pi fkT}$.

Hence,

$$X(f)=P(f)\left(m[0]+m[1]e^{-j1\pi fT}+m[2]e^{-j2\pi fkT}+m[3]e^{-j2\pi fkT}\right);\ell=4$$

$$=\frac{A}{j2\pi f}(1-z)\left(m[0]+m[1]z+m[2]z^2+m[3]z^3\right);\ z=e^{-j1\pi fT}$$

$$=\frac{A}{j2\pi f}(1-z)\left(1-z+z^2+z^3\right)$$

$$=\frac{A}{j2\pi f}\left(1-2z-2z^2-z^4\right)$$

$$=\frac{A}{j2\pi f}\left(1-2e^{-j1\pi fT}-2e^{-j2\pi fT}-e^{-j4\pi fT}\right)$$

c. Assume T = 2 [ms] and A = 1 [mV]. Find $X(0)$ for the following *m*.

i. $m = (1)$

ii. $m = (1,1)$

iii. $m = (1,1,0,0)$

iv. $m = (1,1,-1)$

v. $m = (1,1,-1,1)$

vi. $m = (1,1,-1,-1)$

vii. $m = (1,1,-1,1,-1,-1,1,1,1,-1,1,1)$

Solution

First, we find

$$X(0) = \int_{-\infty}^{\infty} c(t) e^{-j2\pi 0 t} dt = \int_{-\infty}^{\infty} c(t) dt = AT$$

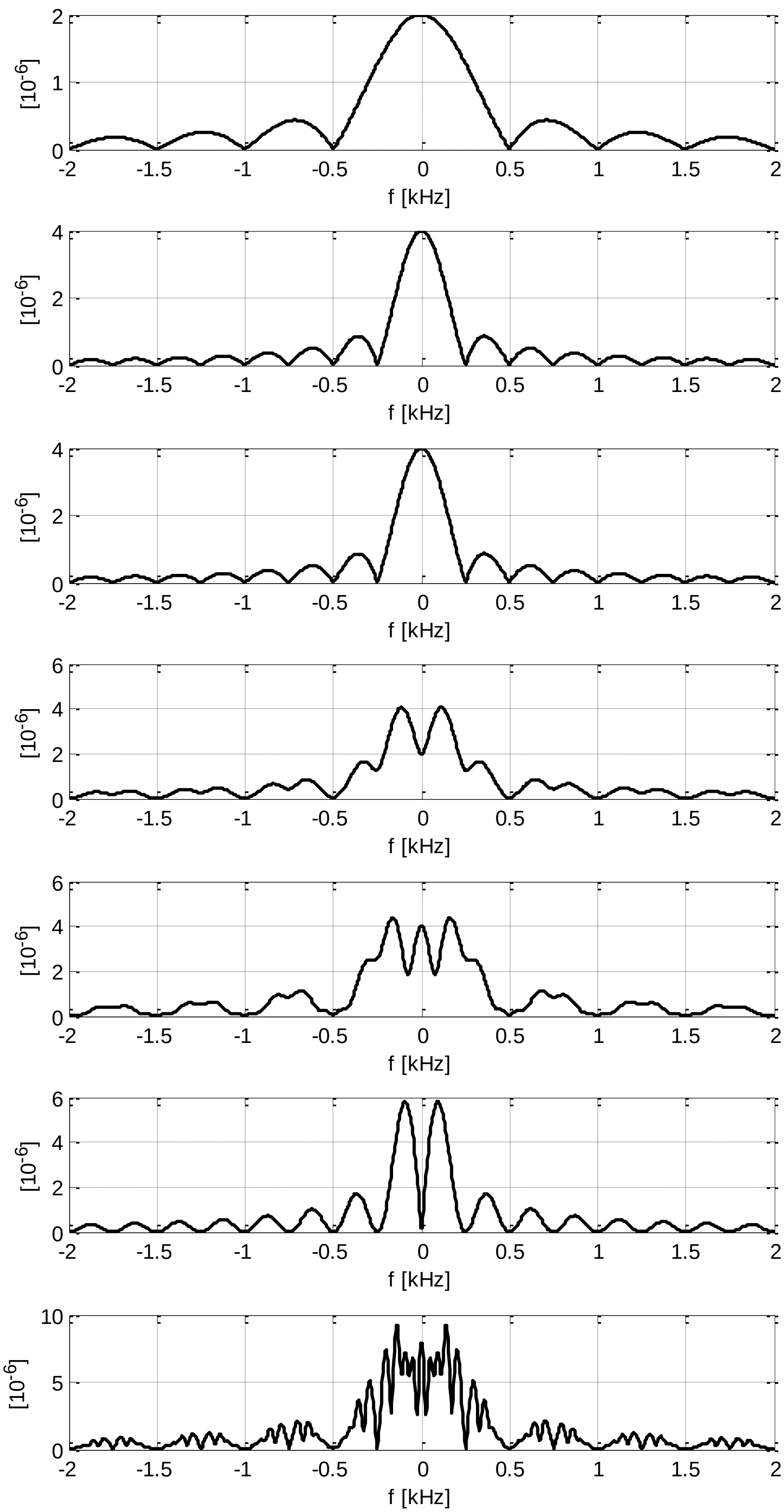
Then

$$X(0) = P(0) \sum_{k=0}^{\ell-1} m[k] e^{-j2\pi 0 kT} = P(0) \sum_{k=0}^{\ell-1} m[k] = AT \sum_{k=0}^{\ell-1} m[k]$$

After plugging in the numbers, we have

$$X(0) \;=\; 2,\; 4,\; 4,\; 2,\; 4,\; 0,\; 8 \quad \times 10^{-6} \quad [\text{V/Hz}]$$

d. All the plots for $|X(f)|$ are shown on the next page



2
1
0
[10-6]
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
f [kHz]
4
2
0
[10-6]
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
f [kHz]
4
2
0
[10-6]
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
f [kHz]
6
4
2
0
[10-6]
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
f [kHz]
6
4
2
0
[10-6]
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
f [kHz]
6
4
2
0
[10-6]
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
f [kHz]
10
5
0
[10-6]
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
f [kHz]

Q2 Poisson Sum Formula [M2011 Q5]

Monday, August 06, 2012
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(a) and (b) Recall that

$$Ш_{T_0}(t) = \sum_k \delta(t - kT_0) \xrightarrow{\mathcal{F}} \frac{1}{T_0}\sum_k \delta(f - kf_0) = f_0 Ш_{f_0}(f)$$

where $f_0 = \frac{1}{T_0}$.

Of course, you may not remember the above fact.
However, I asked you to remember one special case which is much easier to remember:

$$Ш_1(t) \xrightarrow{\mathcal{F}} Ш_1(f)$$

In other words

$$\sum_n \delta(t-n) \xrightarrow{\mathcal{F}} \sum_k \delta(f-k)$$

Let this be $y(t)$.

The special case can be turned into the general case via the scaling property:

From $y(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} Y\left(\frac{f}{a}\right)$, we have

$$\sum_n \delta(at-n) = \frac{1}{|a|}\sum_n \delta\left(t - \frac{n}{a}\right)$$

$$\frac{1}{|a|}\sum_k \delta\left(\frac{f}{a} - k\right) = \frac{|a|}{|a|}\sum_k \delta(f - ak)$$

Recall that $\delta(at) = \frac{1}{|a|}\delta(t)$

Therefore

$$\sum_n \delta\left(t - \frac{n}{a}\right) \xrightarrow{\mathcal{F}} |a| \sum_k \delta(f - ak)$$

Let $a = \frac{1}{T_0}$. We then have

$$\sum_n \delta(t - nT_0) \xrightarrow{\mathcal{F}} \frac{1}{T_0}\sum_k \delta\left(f - \frac{k}{T_0}\right)$$

In this question, this property is applied to $\sum_\ell \delta(t - \ell T)$ to get

$$\sum_{\ell} \delta(t-\ell T) \xrightarrow{\mathcal{F}} \frac{1}{T} \sum_{\ell} \delta\left(f - \frac{\ell}{T}\right)$$

So, by the convolution-in-time rule, we have

$$x(t) \xrightarrow{\mathcal{F}} G(f) \times \frac{1}{T} \sum_{\ell} \delta\left(f + \left(-\frac{\ell}{T}\right)\right)$$

(c) and (d)

The integral under consideration is

$$\int_{-\infty}^{\infty} \underbrace{e^{j2\pi f t} G(f)}_{\text{call this } b(f)} \delta\left(t - \frac{\ell}{T}\right) df$$

By the sifting property of δ-function,

$$\int_{-\infty}^{\infty} b(f) \delta\left(f - \frac{\ell}{T}\right) df = b\left(\frac{\ell}{T}\right) = e^{j2\pi \frac{\ell}{T} t} G\left(\frac{\ell}{T}\right)$$

Summary :

$a = \frac{1}{T}$, $b = -\frac{\ell}{T}$, $c = j2\pi \frac{\ell}{T} t$, $d = \frac{\ell}{T}$

Q3 QAM - Key Equations

Sunday, August 05, 2012
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Let $v(t) = x_{QAM}(t)\sqrt{2}\sin(2\pi f_c t)$

$= m_1(t)\, 2\cos(2\pi f_c t)\sin(2\pi f_c t)$

$+ m_2(t)\, 2\sin(2\pi f_c t)\sin(2\pi f_c t)$

$= m_1(t)\sin(2\pi(2f_c)t) + m_2(t)(1-\cos(2\pi(2f_c)t))$

$\sin^2 x = \left(\frac{e^{jx}-e^{-jx}}{2j}\right)^2 = \frac{e^{2jx}+e^{-2jx}-2}{-4}$

$= \frac{1}{2}(1-\cos(2x))$

$= m_2(t) + \underbrace{m_1(t)\sin(2\pi(2f_c)t)} - \underbrace{m_2(t)\cos(2\pi(2f_c)t)}$

the spectrum is centered around $\pm 2f_c$ — the spectrum is centered around $\pm 2f_c$

$LPF\{v(t)\} = m_2(t) + 0 + 0 = m_2(t)$

Assumption:

If $m_1(t)$ is band limited to B_1

$m_2(t)$ is band limited to B_2,

we need $f_c > B_1$ and $f_c > B_2$